RATIONALITY

A **GAME** is a process consisting in:
- a set of players
- an initial situation
- rules that players must follow
- all possible final situations
- the preferences of all players

Note that players are selfish: only care about own preferences
and rational: see later

* PREFERENCE RELATION

Let X be a set. A preference relation on X is a binary relation $\succeq$ s.t. for all $x, y, z \in X$:

- $x \succeq x$ — REFLEXIVE
- $x \succeq y$ or $y \succeq x$ or both — COMPLETE (ordine)
- $x \succeq y$ and $y \succeq z \Rightarrow x \succeq z$ — TRANSITIVE

** UTILITY FUNCTION

Let $\succeq$ be a preference relation over X. A utility function representing $\succeq$ is a function $u: X \to \mathbb{R}$ s.t.

$$u(x) \geq u(y) \iff x \succeq y$$

RATIONALITY ASSUMPTIONS

ASSUMPTION 1:

The players are able to provide a preference relation* over the outcomes of the game.

ASSUMPTION 2:

The agents of the game (players) are able to provide a utility function** representing their preferences relations, whenever necessary.

ASSUMPTION 3:

The players use consistently the probability laws, in particular when computing the expected utility.

ASSUMPTION 4:

The players are able to understand and analyze the consequences of all their actions, the consequences of this information on any other player, the consequences of the consequences and so on.

ASSUMPTION 5:

The players are able to use decision theory, whenever possible, that is given a set of alternatives X, and a utility function u on X, each player seeks an $\bar{x} \in X$ s.t.

$$u(\bar{x}) \geq u(x) \quad , \forall x \in X$$

EXTENSIVE FORM

- the moves are in sequence
- every possible situation is known to the players, at any time they know the whole past history and the possible developments.

Note that this game is a game with perfect information
Utility is needed only when probability is involved.

FINITE DIRECTED GRAPH

A pair (V,E) where
- V is a finite set (vertices)
- $E \subset V \times V$ is a set of ordered pairs of vertices (directed edges)

A path from a vertex v_1 to a vertex v_{k+1} is a finite sequence of vertices-edges $v_1, e_1, \dots, v_k, e_k, v_{k+1}$ s.t. $e_i \neq e_j$ if $i \neq j$

The length of a game is the length of the longest path in the game.

An oriented graph is a finite directed graph having no bidirected edges

A tree is a triple (V, E, x_0) where (V,E) is an oriented graph and x_0 is a vertex in V s.t. there is a unique path from x_0 to x_i, $\forall x_i \in V$

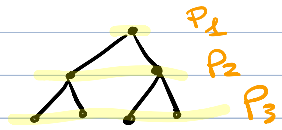
A child of a vertex v is any vertex x s.t. $(v,x) \in E$.
A vertex is called a leaf if it has no children.
A vertex x follows vertex v if there is a path from v to x.

GAME IN EXTENSIVE FORM

1) A finite set $N = \{1 \dots n\}$ of players
2) A game tree (V, E, x_0)
3) A partition $\{P_1, \dots, P_{n+1}\}$ of the vertices which are not leaves.
   $\rightarrow P_i \cap P_j = \emptyset$, $P_1 \cup \dots \cup P_{n+1} = V$
4) A probability distribution for each vertex in P_{n+1}, defined on the edges from the vertex to its children
5) An n-dimensional vector attached to each leaf $\Rightarrow$ utilities

OBS

- Note that the set P_i for $i \leq n$ is the set of the nodes v where player i must choose a child of v, representing a possible move for them.



- P_{n+1} is the set of the nodes where a chance move is present. P_{n+1} can be empty
- When P_{n+1} is empty (no chance) the players need to have only preferences on the leaves: a utility function is not required

BACKWARD INDUCTION

Decision theory allows to solve games of length 1. Assumption 4 allows to solve a game of length $i+1$ if the games of length at most i are solved. Thus we can solve games of any finite length.

> **Theorem**
> The rational outcomes of a finite, perfect information game are those given by the procedure of backward induction.

Note that uniqueness is not guaranteed.

> **Theorem (chess – Von Neumann)**
> In the game of chess one and only one of the following holds:
> - the white has a way to win, no matter what the black does.
> - the black has a way to win, no matter what the white does.
> - the white (black) has a way to force at least a draw, no matter what the black (white) does.

Proof

Suppose the length of the game is $2k$, so each player has k choices to make. a_i = move of white at ith stage, b_i = move black at ith stage.

"White has a winning strategy, no matter what black does" can be expressed as:

$$\exists a_1 : \forall b_1 \exists a_2 : \forall b_2 \dots \exists a_k : \forall b_k \Rightarrow \text{white wins}$$

Suppose this weren't true, then

$$\forall a_1 \exists b_1 : \forall a_2 \exists b_2 : \dots \forall a_k \exists b_k \Rightarrow \text{white does not win}$$

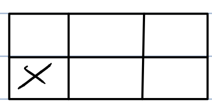
This means black can obtain at least a draw. → Symmetrically for black → If the first two aren't true, then the last is.

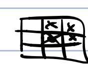
Corollary (very obvious: if there is no possibility to tie....)

Consider a finite perfect information game with two players, where the possible outcomes are the victory of one or the other player. Then one and only one of the following holds:
- the first player can win, no matter what the other does
- the second player can win, no matter what the other does

Very weak solution: the game has a rational outcome, but it is inaccessible (chess)
Weak solution: the outcome of the game is known, but how to get it is not (general)
Solution: it is possible to provide an algorithm to find a solution

CHOMP

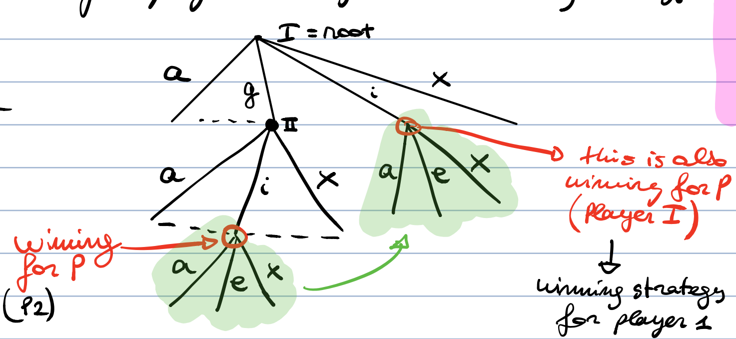


the one who removes the last X square loses.
If you remove a square, also all the ones above and to the right of it disappear. 

In a finite chomp game the first player always has a winning strategy.

Proof:

a	~~b~~	~~c~~	~~d~~
e	~~f~~	~~g~~	~~h~~
X	~~j~~	~~j~~	~~k~~



In general, assume P_2 has a winning strategy against any of P_1 initial moves. If a winning response existed for P_2, then P_1 could have played it as their first move and thus forced victory.

NOTE: P_2 can only win in $2 \times \infty$

An impartial combinatorial game is a game s.t.

1) There are two players moving in alternate order.
2) There is a finite number of positions in the game
3) Both players follow the same rules.
4) The game ends when no further moves are possible
5) The game does not involve chance
6) In the classical version the winner is the player leaving the other player with no available moves, in the misère version the opposite.

P-positions : win for Previous (if you are in P → lose)
N-positions : win for Next (if you are in N → win)

- terminal positions are P-positions
- from a P-position only N-positions are available
- from an N-position it is possible to go to a P-position

NIM GAME

Nim Game is defined as $(n_1, \dots, n_k)$ where for all i n_i is a positive integer. A player at their turn has to take one and only one n_i and substitute it with $\hat{n}_i < n_i$. The winner is the player arriving at the position $(0 \dots 0)$.

The game of taking away cards from one pile. Goal: clear the table

HOW TO SOLVE THE NIM GAME

Define an operation $\oplus$ on $\mathbb{N}$

1) write n_1, n_2 in binary form
2) write the sum $[n_1]_2 \oplus [n_2]_2$ in binary form where $\oplus$ is sum without carry.
3) the result is the obtained number, written in binary form.

EX.

$$\begin{array}{ccc} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \\ \hline 1 & 1 & 0 \end{array}$$

A non empty set A with an operation $\cdot$ on it is called a group provided:

- for $a, b \in A$, the element $a \cdot b \in A$
- $\cdot$ is associative : $(a \cdot b) \cdot c = a \cdot (b \cdot c)$
- there is an element e (identity) such that $a \cdot e = e \cdot a = a$, $\forall a \in A$
- for every $a \in A$, there is $b \in A$ such that $a \cdot b = b \cdot a = e$. Such element is called inverse of a.

If $a \cdot b = b \cdot a$, $\forall a, b \in A$, the group is called Abelian.

The cancelation law holds: $a \cdot b = a \cdot c \Rightarrow b = c$

The set of natural numbers with $\oplus$ is an abelian group.

Theorem - Bouton
A $(m_1, \dots, m_K)$ position in the Nim game is a P-position iff $m_1 \oplus m_2 \oplus \dots \oplus m_K = 0$

Proof

- terminal states are P-positions → by definition who gets to $(0, \dots, 0)$ is the winner.
- positions s.t. $m_1 \oplus \dots \oplus m_N = 0$ go away to positions with Nim sum $\neq 0$ → if this weren't to be so, then the new position would be $m_i' \oplus \dots \oplus m_N = 0 = m_i \oplus \dots \oplus m_N$, then $m_i' = m_i$ by the cancellation law, which is impossible as something must have changed
  only 1 changes b/c that's the rule of the game
- positions s.t. $m_i \oplus \dots \oplus m_N \neq 0$ can go to positions with $m_i' \oplus \dots \oplus m_N = 0$
  Let $z := m_i \oplus \dots \oplus m_N \neq 0$. Take a pile having 1 in the leftmost column where the expansion of z has 1, put there 0 and go right, leaving unchanged a digit corresponding to a 0 in the expansion of the sum, changing it otherwise. It is easy to check that the result is smaller than the original number

STRATEGIES

In backward induction a move must be specified at any node. P_i is the set of the nodes where i is called to make a move.

A pure strategy for player i is a function defined on the set P_i, associating to each node v in P_i a child w, or equivalently an edge (v, x).
A mixed strategy is a probability distribution on the set of the pure strategies

When a player has n pure strategies, the set of their mixed strategies is
$$\Sigma_n = \{ p = (p_1, \dots, p_n) : p_i \geq 0 ,\ \sum p_i = 1 \}$$
Σ_n is the fundamental simplex in n-dimensional space.

NOTE
If $P_i = \{ v_1, \dots, v_k \}$ and v_j has n_j children, then the number of strategies of player i is $n_1 \cdot \dots \cdot n_k$

GAMES WITH IMPERFECT INFORMATION

An information set for a player i is a pair $(U_i, A(U_i))$ with the following properties:
- $U_i \subset P_i$ is a nonempty set of vertices $v_1, \dots, v_k$.
- Each $v_j \in U_i$ has the same number of children.
- $A_i(U_i)$ is a partition of the children of $v_1 \cup \dots \cup v_k$ with the property that each element of the partition contains exactly one child of each vertex v_j.
  → each set of the partition (that is each children) represents an available move for the player

An extensive form game with imperfect information is constituted by:
- A finite set of players N
- A game tree (V, E, x_0)
- A partition made by sets $P_1, \dots, P_{n+1}$ of the vertices which are not leaves
- A partition (U_i^j), $j = 1 \dots k_i$, of the set P_i, for all i, with (U_i^j, A_i^j) information set for all i for all j.
- A probability distribution, for each vertex in P_{n+1}, defined on the edges going from the vertex to its children.
- An n-dimensional vector attached to each leaf.

A pure strategy for player i in an imperfect information game is a function defined on the collection U of his information sets and assigning to each U_i in U an element of the partition $A(U_i)$.
A mixed strategy is a probability distribution over the pure strategies.

THE NASH MODEL

Non Cooperative Game

A two player non cooperative game in strategic form is
$(X, Y, f: X \times Y \to \mathbb{R} \ ; \ g: X \times Y \to \mathbb{R})$
X, Y are the strategy sets of the players, f, g are their utilities.

Pure Strategy

When a player chooses one strategy with probability 1

Mixed Strategy

When a player chooses a strategy with probability < 1
given $M \in \mathbb{N}$ strategies $(x_1, x_2, \dots, x_M)$, choose M probabilities p_i s.t.
$p_i \geq 0$ and $\sum_{i=1}^{M} p_i = 1$. There are $\mathbb{R}^M$ possible choices.

Strategies

$\underset{M}{X}, \underset{N}{Y} \qquad f, g: X \times Y \to \mathbb{R}$

$\hat{X}, \hat{Y} \quad p \in \hat{X} \qquad p_i \geq 0 \quad \sum_i p_i = 1$
$\qquad\quad q \in \hat{Y} \qquad q_i \geq 0 \quad \sum_i q_i = 1$

Expected utility $= \sum_{i,j}^{M,N} p_i q_j \, f(x_i, y_j) = \hat{f}(p, q)$

define a matrix $A \in M \times N \to A_{ij} = f(x_i, y_j)$
define a matrix $B \in M \times N \to B_{ij} = g(x_i, y_j)$
} matrices of utilities.

Expected utility $= p^T A q$

vettori riga

Nash equilibrium profile

A NE profile for the $(X, Y, f: X \times Y \to \mathbb{R}, g: X \times Y \to \mathbb{R})$ non-cooperative game
is a pair $(\bar{x}, \bar{y}) \in X \times Y$ s.t.

$$f(\bar{x}, \bar{y}) \geq f(x, \bar{y}) \quad \forall x \in X$$
$$g(\bar{x}, \bar{y}) \geq g(\bar{x}, y) \quad \forall y \in Y$$

A NE profile is a joint combination of strategies, stable w.r.t. unilateral deviations of a single player.

(Weakly) dominant

$\bar{x}$ is a (weakly) dominant strategy if $f(\bar{x}, y) \geq f(x, y)$ for all x, y.

Theorem

If $\bar{x}$ is a (weakly) dominant strategy for P_1, then if $\bar{y}$ maximizes the function $y \to g(\bar{x}, y)$ [that is the utility of P_2], $(\bar{x}, \bar{y})$ is a NEp.

Backward induction provides a NEp for a game of perfect information. Is it possible that in games of perfect information there are more than those provided by backward induction? $\longrightarrow$ ~~NO~~ YES Consider the game I have 1 and offer you $1-x$. If you accept then I get x and you get $1-x$, otherwise both get 0.

Backward Induction $\longrightarrow (1,0)$ VS NE $\longrightarrow (x, 1-x)$

	A	R
A	5,0	0,0
B	4, 1x	0,0 x
C	3, 2x	0,0 x
D	2, 3x	0,9 x
E	1, 4x	0,0 x
F	0,5 x	0,0 x

Best Response

Denote by BR the following multifunction

$BR_1 : Y \rightarrow X : BR_1(y) = \arg\max_x (f(x,y))$

$BR_2 : X \rightarrow Y : BR_2(x) = \arg\max_y (g(x,y))$

and

$$BR : X \times Y \rightarrow Y \times X : BR(x,y) = (BR_1(y), BR_2(x))$$

Theorem

$(\bar{x},\bar{y})$ is a NEp for the game iff:

$(\bar{x},\bar{y}) \in BR(\bar{x},\bar{y})$

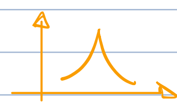
Thus the existence of a NEp in a game is equivalent to the existence of a fixed point for the BR function.

NASH Theorem $\longrightarrow$ This is 2 players case, below is n-player

Given the game $(X, Y, f: X\times Y \rightarrow \mathbb{R}, g: X \times Y \rightarrow \mathbb{R})$, a finite set is NOT convex

If:

1) X and Y are compact (close and bounded) convex (given 2 points, the segment that connects them is entirely contained in the set) subsets of some Euclidean space
2) f, g continuous
3) $x \mapsto f(x,y)$ is (quasi) concave $\forall y$
4) $y \mapsto g(x,y)$ is (quasi) concave $\forall x$



Quasi concavity for a real valued function h means that the sets $h_a = \{z : h(z) \geq a\}$ are all convex for all a (maybe empty for some a)

Then the game has at least one NEp.

COROLLARY $\longrightarrow$ This is 2 players case, below is n-player

A finite game (A,B) admits always a NEp in mixed strategies

Once fixed the strategies of the other players, the utility function of one player is linear with its own variable.

Proof

In this case X and Y are simplexes, $f(x,y) = x^t A y$, $g(x,y) = x^t B y$ $\longrightarrow$ the assumptions of the NE theorem are fulfilled

Note that strategies can be either pure or mixed, and pure ones are just a specific case of the mixed ones (probability 1 for an action and 0 for all the others). So of course every finite game admits an NEp in mixed strategies.

's enough?

INDIFFERENCE

Suppose $(\bar{x},\bar{y})$ is a NE in mixed strategies. Suppose $\text{spt}\,\bar{x} = \{1,..,k\}$, $\text{spt}\,\bar{y} = \{1,...,\ell\}$ and $f(\bar{x},\bar{y}) = V$. Then:

support

$\{i : \bar{x}_i > 0\}$

$$\begin{cases} a_{11}\bar{y}_1 + \dots + a_{1\ell}\bar{y}_\ell = V \\ \vdots \\ a_{k1}\bar{y}_1 + \dots + a_{k\ell}\bar{y}_\ell = V \\ a_{(k+1)1}\bar{y}_1 + \dots + a_{(k+1)\ell}\bar{y}_\ell \leq V \\ \vdots \\ a_{m1}\bar{y}_1 + \dots + a_{m\ell}\bar{y}_\ell \leq V \end{cases}$$

optimal (first k rows), suboptimal (remaining rows)

Analo

Theorem – NE / BR n-player

Consider an n-player game with strategy sets X_i and payoffs $f_i : X \to \mathbb{R}$, with $X = \bigtimes_{j=1..n} X_j$

If $x = (x_1, \dots, x_{i-1}, x_i, x_{i+1}, \dots, x_n)$ is a strategy profile,

denote by x_{-i} the vector $x_{-i} = (x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n)$ and $x = (x_i, x_{-i})$

Then

$$\bar{x} = (\bar{x}_i)_{i=1}^{n} \text{ is a NEp iff } \forall i \in N \text{ we have } \bar{x}_i \in BR_i(\bar{x}_{-i})$$

NASH Theorem

Given a n-player game with strategy sets X_i and payoff functions $f_i : X \to \mathbb{R}$ where $X = \prod_{i=1}^{n} X_i$. If:

- each X_i is a closed bounded convex subset in a finite dimensional space $\mathbb{R}^{d_i}$ → dimension of strategy space
- each $f_i : X \to \mathbb{R}$ is continuous
- $x_i \to f_i(x_i, x_{-i})$ is a quasi concave function for each fixed $x_{-i} \in X_{-i}$

} utility function

Then the game admits at least one NEp.

Mixed equilibria for n-player finite games

Consider an n-player finite game with strategy sets A_i and payoffs $f_i(a_1, \dots, a_n)$.
In the mixed extension each player i chooses a probability distribution $x^i \in \Sigma_{A_i}$, that is to say $x^i_{a_i} \geq 0$ for all $a_i \in A_i$ and $\sum_{a_i \in A_i} x^i_{a_i} = 1$

Denote $A = \prod_{i=1}^{n} A_i$ the set of pure strategy profiles. The probability of observing an outcome $(a_1, \dots, a_n) \in A$ is the product $\prod_{i=1}^{n} x^i_{a_i}$ and the expected payoffs are:

$$\bar{f}_i(x^1, \dots, x^n) = \sum_{(a_1, \dots, a_n) \in A} f_i(a_1, \dots, a_n) \cdot \prod_{j=1}^{n} x^j_{a_j} = \sum_{a_i \in A_i} x^i_{a_i} \, v_i(a_i, x^{-i})$$

$$v_i(a_i, x^{-i}) = \sum_{a_j \in A_j,\, j \neq i} f_i(a_1, \dots, a_n) \cdot \prod_{j \neq i} x^j_{a_j}$$

Corollary

Every n-player finite game has at least one NEp in mixed strategies.

Congestion Games

A congestion game is defined as:

- N: players
- R: a set of resources
- A collection of subsets of R (strategies)
- for each r, a function $d_r : \mathbb{N} \to \mathbb{R}$ (cost of using resource r), # of players using $r \mapsto$ cost

Given $s = (s_1, \dots, s_n)$ strategy profiles for players, define $x(r, s)$ as the number of players s.t. $r \in s_i$.
The cost for player i is

$$c_i(s_1, \dots, s_n) = \sum_{r \in s_i} d_r(x(r, s))$$

A player can choose which resources to use. The cost to use a resource depends on the number of people using it.

Duopoly Models

Two firms choose quantities of a good to produce. Firm_1 produces q_1, firm_2 produces q_2. The unitary cost is $c>0$ for both. A quantity $a>c$ of the good saturates the market. The price $p(q_1,q_2)$ is $p=\max\{a-(q_1+q_2),0\}$

Payoffs: $u_1(q_1,q_2)=q_1\cdot p(q_1,q_2)-cq_1=q_1(a-(q_1+q_2))-cq_1$

$u_2(q_1,q_2)=q_2\cdot p(q_1,q_2)-cq_2=q_2(a-(q_1+q_2))-cq_2$

Monopoly

Suppose $q_2=0$ // just one firm in the market

HIGHER UTILITY

Then firm_1 maximizes $u_1=q_1(a-q_1)-cq_1 \longrightarrow a-q_1-q_1-c=0 \longrightarrow q_M=\frac{a-c}{2}$, $p_M=\frac{a+c}{2}$, $u_M(q_M)=\frac{(a-c)^2}{4}$

Duopoly

$u_1=q_1(a-(q_1+q_2))-cq_1 \longrightarrow a-q_1-q_2-q_1-c=0 \longrightarrow q_1=\frac{-q_2+a-c}{2}$

$u_2=q_2(a-(q_1+q_2))-cq_2 \longrightarrow a-q_2-q_1-q_2-c=0 \longrightarrow q_2=\frac{-q_1+a-c}{2}$

LOWER PRICE

$q_1=\frac{\frac{q_1-a+c}{2}+a-c}{2} \longrightarrow 4q_1=q_1-a+c+2a-2c \longrightarrow q_1=\frac{a-c}{3}=q_2$

×2

HIGHER QUANTITY

$p=a-(q_1+q_2)=a-2q_1=a-\frac{2a-2c}{3}=\frac{a+2c}{3}$

$u_1(q_1)=q_1\overbrace{(a-(q_1+q_2))}^{p}-cq_1=\frac{a-c}{3}\cdot\frac{a+2c}{3}-c\cdot\frac{a-c}{3}=\frac{a^2+2ac-ac-2c^2-3ac+3c^2}{9}=$

$=\frac{(a-c)^2}{9}$

Leader

, one of the two firms is the leader, the other is the follower

$\bar{q}_2(q_1)=\frac{a-q_1-c}{2}$ // firm_2 maximing utility.

The leader maximizes $u_1(q_1,\bar{q}_2)=u_1\left(q_1,\frac{a-q_1-c}{2}\right)=q_1\left(a-\left(q_1+\frac{a-q_1-c}{2}\right)\right)-cq_1=q_1\left(a-\frac{q_1+a-c}{2}\right)-cq_1=$

$=aq_1-\frac{q_1}{2}(q_1+a-c)-cq_1 \longrightarrow a-\frac{1}{2}(q_1+a-c)-\frac{q_1}{2}-c=0 \longrightarrow \bar{q}_1=\frac{a-c}{2}$

$\bar{q}_1=\frac{a-c}{2}$, $\bar{q}_2=\frac{a-\bar{q}_1-c}{2}=\frac{a-\frac{a-c}{2}-c}{2}=\frac{a-c}{4}$

$u_1(\bar{q}_1,\bar{q}_2)=\frac{(a-c)^2}{8}$, $u_2(\bar{q}_1,\bar{q}_2)=\frac{(a-c)^2}{16}$, $p=a-(\bar{q}_1+\bar{q}_2)=a-\frac{a-c}{2}-\frac{a-c}{4}=\frac{a+3c}{4}$

ZERO SUM GAMES

A two player zero sum game in strategic form is the triplet $(X, Y, f: X \times Y \to \mathbb{R})$

Representing the game with a payoff matrix P

$$P = \begin{pmatrix} 4 & 3 & 1 \\ 7 & 5 & 8 \\ 8 & 2 & 0 \end{pmatrix} \quad , \ i \text{ rows}, \ j \text{ columns}$$

Player 1 is guaranteed to obtain at least $V_1 = \max_i \left(\min_j p_{ij} \right)$ // P_1 can obtain min V_1

Player 2 is guaranteed to pay no more than $V_2 = \min_j \left(\max_i p_{ij} \right)$ // P_2 can obtain max V_2

In general
$$\left.\begin{array}{l} V_1 = \sup_x \overbrace{\inf_y f(x,y)}^{\alpha(x)} \\ V_2 = \inf_y \underbrace{\sup_x f(x,y)}_{\beta(y)} \end{array}\right\} \text{conservative values of } P_1 \text{ and } P_2$$

OPTIMALITY (for $\alpha(x)$, $\beta(y)$ see slides page 8)

Suppose:
- $V_1 = V_2 = V$
- there exists strategy $\bar{x}$ s.t. $f(\bar{x}, y) \geq v$ for all $y \in Y$
- there exists strategy $\bar{y}$ s.t. $f(x, \bar{y}) \leq v$ for all $x \in X$

Then:
- v is the rational outcome of the game
- $\bar{x}$ is an optimal strategy for P_1
- $\bar{y}$ is an optimal strategy for P_2

Observe:
- $\bar{x}$ is optimal for P_1 as it maximizes $\alpha(x) = \inf_y f(x,y)$, that is the value of the optimal choice of P_2
- $\bar{y}$ is optimal for P_2 as it minimizes $\beta(y) = \sup_x f(x,y)$, that is the value of the optimal choice of P_1

> Let X, Y be nonempty sets and let $X \times Y \to \mathbb{R}$ be an arbitrary real valued function. Then $\sup_x \inf_y f(x,y) \leq \inf_y \sup_x f(x,y)$, that is $V_1 \leq V_2$

Proof

$\inf_y f(x,y) \leq f(x,y) \leq \sup_x f(x,y)$, thus $\alpha(x) \leq \beta(y)$ and consequently $\sup_x \alpha(x) \leq \inf_y \beta(y)$ $\blacksquare$

To prove existence of a rational outcome:
- $V_1 = V_2$
- there exists $\bar{x}$ fulfilling $V_1 = \inf_y f(\bar{x}, y)$
- there exists $\bar{y}$ fulfilling $V_2 = \sup_x f(x, \bar{y})$

Theorem - Von Neumann

A two player, finite, zero sum game described by a payoff matrix P has a rational outcome

DUAL PROBLEMS

Player 1:
$$\begin{cases} \max_{x,v} \ v \\ P^t x \geq v \cdot I_m \\ x \geq 0 \end{cases} \qquad \text{DUAL}$$

Player 2:
$$\begin{cases} \min_{y,w} \ w: \\ P y \leq w I_n \\ y \geq 0 \end{cases} \qquad \text{PRIMAL}$$

Theorem - Duality

Let v be the value of the primal min problem and V the value of the dual max problem. Then $v \geq V$

Theorem - Strong Duality

If the primal and dual problems are feasible, then both problems have optimal solutions $\bar{x}, \bar{y}$ and the optimal values coincide $\to v = V$ $\to$ no duality gap

- If the primal is feasible and the dual is infeasible $\to v = V = -\infty$
- If the primal is infeasible and the dual is feasible $\to v = V = \infty$
- If both are infeasible $\to v = \infty, V = -\infty$

COROLLARY

If one problem is feasible and has an optimal solution, then also the dual problem is feasible and has solutions. Moreover there is no duality gap.

COMPLEMENTARITY CONDITIONS

$$P \begin{cases} \min c^t x \\ Ax \geq b, \; x \geq 0 \end{cases} \qquad D \begin{cases} \max b^T y \\ A^T y \leq c, \; y \geq 0 \end{cases}$$

Let $\bar{x}, \bar{y}$ be primal and dual feasible. Then $\bar{x}, \bar{y}$ are simultaneously optimal iff

$$CC \begin{cases} (\forall i = 1, \ldots, n) & \bar{x}_i > 0 \Rightarrow \sum_{j=1}^{m} a_{ji} \bar{y}_j = c_i \\ (\forall j = 1, \ldots, m) & \bar{y}_j > 0 \Rightarrow \sum_{i=1}^{n} a_{ij} \bar{x}_i = b_j \end{cases}$$

$$\Downarrow$$

$$CC \begin{cases} (\forall i = 1, \ldots, n) & \bar{x}_i > 0 \Rightarrow \sum_{j=1}^{m} p_{ji} \bar{y}_j = V \\ (\forall j = 1, \ldots, m) & \bar{y}_j > 0 \Rightarrow \sum_{i=1}^{n} p_{ij} \bar{x}_i = V \end{cases}$$

- Since $\bar{y}$ is optimal for player 2, he is able to pay no more than v against all strategies of the first player.
- If $\bar{x}_i > 0$, then player 1 plays row i with positive probability.
  The complimentary condition shows then that row i must be optimal for player 1.

SUMMARY

- A finite zero sum game has always a rational outcome in mixed strategies.
- The set of optimal strategies for the players is a nonempty closed convex set.
- The outcome, at each pair of optimal strategies, is the common conservative value v of the players.

Theorem

Let X, Y be sets and $f: X \times Y \to \mathbb{R}$ a function. Then the following are equivalent:

1) The pair $(\bar{x}, \bar{y})$ fulfills $f(x,\bar{y}) \leq f(\bar{x},\bar{y}) \leq f(\bar{x},y)$ $\forall x \in X$, $\forall y \in Y$

2) The following conditions are satisfied:

(i) $\inf_y \sup_x f(x,y) = \sup_x \inf_y f(x,y)$

(ii) $\inf_y f(\bar{x},y) = \sup_x \inf_y f(x,y)$

(iii) $\sup_x f(x,\bar{y}) = \inf_y \sup_x f(x,y)$

Proof

$1 \Rightarrow 2$ From 1) we get $V_2 = \inf_y \sup_x f(x,y) \leq \sup_x f(x,\bar{y}) = f(\bar{x},\bar{y}) = \inf_y f(\bar{x},y) \leq \sup_x \inf_y f(x,y) = V_1$

But since $V_1 \leq V_2$, and we just showed $V_2 \leq V_1$, then $V_1 = V_2$, thus all above ineq. are equalities.

$2 \Rightarrow 1$ From 2) we get $\inf_y \sup_x f(x,y) \overset{(iii)}{=} \sup_x f(x,\bar{y}) \geq f(\bar{x},\bar{y}) \geq \inf_y f(\bar{x},y) \overset{(ii)}{=} \sup_x \inf_y f(x,y)$

But since (i) holds then all the above ineq. are equalities ∎

PROPOSITION

Any $(\bar{x},\bar{y})$ NE of the zero sum game provides optimal strategies for the players.
Any pair of optimal strategies for the players provides a NE for the zero sum game.

Thus Nash Theorem generalites Von Neumann.

DEFINITIONS

ANTISYMMETRIC a square matrix $n \times n$ $P = p_{ij}$ is said to be a. provided $p_{ij} = -p_{ji}$ for all $i,j = 1 \dots n$

FAIR a game is said to be fair if the associated matrix is antisymmetric.

PROPOSITION

If $P = p_{ij}$ is antisymmetric the value is 0 and $\bar{x}$ is an optimal strategy for Player 1 iff it is optimal for Player 2.

Proof

Since $x^t P x = (x^t P x)^t = x^t P^t x = -x^t P x$

$f(x,x) = 0$ $\forall x$, thus $V_1 \geq 0$, $V_2 \leq 0$ $\longrightarrow$ $V = 0$

If $\bar{x}$ is optimal for the first player, $\bar{x}^t P y \geq 0$ $\forall y$.

Transposing $\rightarrow$ $y^T P \bar{x} \leq 0$ $\forall y$ $\rightarrow$ thus $\bar{x}$ is optimal also for the second player, and vice versa ∎

REPEATED GAMES

Consider the following game:

$$\begin{pmatrix} 3,3 & 0,10 \\ 10,0 & 1,1 \end{pmatrix} \longrightarrow NE = (1,1)$$

We would like to show that if the game is played a sufficiently large number of times, the players can get at least $3-a$ each, on average
$a>0$

We say the game is played once a day for N days
Each player uses the first strategy for the first $N-k$ days, and the second for k.

Therefore, each player gets : $\frac{(N-k)\cdot 3 + k\cdot 1}{N}$ each day $\longrightarrow$ this is a NE, see slide 6 for proof.

$$\lim_{N\to\infty} \frac{(N-k)\cdot 3 + k\cdot 1}{N} = 3 \quad \blacksquare$$

CORRELATED EQUILIBRIA

Consider the game

$$\begin{array}{c|cc} & F & C \\ \hline F & 2,1 & 0,0 \\ C & 0,0 & 1,2 \end{array} \longrightarrow NE : (2,1), (1,2), \left(\frac{2}{3}, \frac{2}{3}\right)$$

mixed

IDEA: what if the players chose to play (F,F) if a tossed coin is heads and (C,C) if it is tails?

$$U = \frac{1}{2}\cdot 2 + \frac{1}{2}\cdot 1 = 1 + \frac{1}{2} = \frac{3}{2} \longrightarrow \left(\frac{3}{2}, \frac{3}{2}\right)$$

CORRELATED EQUILIBRIUM

Consider the game $(A,B) = (a_{ij}, b_{ij})$, $i = 1, \dots, m$, $j = 1, \dots, n$.
Let $I = \{1, \dots, m\}$, $J = \{1, \dots, n\}$ and $X = I \times J$
A correlated equilibrium is a probability distribution $p = (p_{ij})$ on X such that, for all $i' \in I$

$$\sum_{j=1}^{n} p_{ij} a_{ij} \geq \sum_{j=1}^{n} p_{ij} a_{i'j} \qquad \forall i \in I$$

for all $j' \in J$

$$\sum_{i=1}^{m} p_{ij} b_{ij'} \geq \sum_{i=1}^{m} p_{ij} b_{ij} \qquad \forall j \in J$$

AN EXAMPLE

$$\begin{pmatrix} 6,6 & 2,7 \\ 7,2 & 0,0 \end{pmatrix} \qquad \begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix}$$

$$\begin{cases} 6x_1 + 2x_2 \geq 7x_1 & \text{1st ROW} \\ 7x_3 \geq 6x_3 + 2x_4 & \text{2nd ROW} \\ 6x_1 + 2x_3 \geq 7x_1 & \text{1st COL} \\ 7x_2 \geq 6x_2 + 2x_4 & \text{2nd COL} \\ \sum_i x_i = 1 \\ x_i \geq 0 \end{cases}$$

The oracle will tell Player 1 to play 1st row. Player 1 now knows that the oracle's outcome was either x_1 or x_2. So they have to choose between $6x_1 + 2x_2$ and $7x_1 + 0x_2$, since they don't have to follow the oracle's advice, but they know that Player 2 will choose col 1 with probability x_1 or col 2 with x_2

Theorem

A NEp generates a correlated equilibrium $\rightarrow$ the set of the correlated equilibria of a game is nonempty.

Proof

Given a NEP $(\bar{x}, \bar{y})$, the probability distribution on the outcome matrix is $p = (p_{ij})$ with $p_{ij} = \bar{x}_i \bar{y}_j$

We have to prove that $\sum_{j=1}^{m} \bar{x}_{i'} \bar{y}_j a_{i'j} \geq \sum_{j=1}^{m} \bar{x}_{i'} \bar{y}_j a_{ij} \quad \forall i \in I$

If $\bar{x}_{i'} = 0 \rightarrow$ obvious

If $\bar{x}_{i'} \geq 0$ we need to show that $\sum_{j=1}^{m} \bar{y}_j a_{i'j} \geq \sum_{j=1}^{m} \bar{y}_j a_{ij} \quad \forall i \in I$

The left (right) hand side is the expected utility of the first player if he plays row i' (row i) and the second plays $\bar{y}$.

The inequality holds since the pure strategy i' is played with positive probability so i' must be a best reaction to $\bar{y}$.

Theorem

The set of the correlated equilibria of a finite game is a nonempty convex polytope. Proof ....

PROPOSITION

If a row i' is strictly dominated, then $p_{i'j} = 0$ for every j.

Proof

Suppose i' is strictly dominated by i. This implies $(a_{i'j} - a_{ij}) < 0$ for all j.

Since $p_{i'j} \geq 0 \ \forall j$ and $\sum_{j=1}^{m} p_{i'j} a_{i'j} \geq \sum_{j=1}^{m} p_{i'j} a_{ij} \rightarrow \sum_{j=1}^{m} p_{i'j}(a_{i'j} - a_{ij}) \geq 0$, then it must be $p_{i'j} = 0 \ \forall j$ ■

POTENTIAL GAMES

NASH EQUILIBRIUM IN PURE STRATEGIES

Consider a game with strategy sets X_i and suppose that all the players have the same payoff $p: X \to \mathbb{R}$, that is $u_i(x_1, \dots, x_n) = p(x_1, \dots, x_n)$

Take $\bar{x} = (\bar{x}_1, \dots, \bar{x}_n) \in X$ a strategy profile such that $p(\bar{x}) \geq p(x)$ for all strategy profiles $x \in X$. Then $\bar{x}$ is a NE in pure strategies.

PAYOFF EQUIVALENCE

The payoffs $\tilde{u}_i$ and u_i are said diff - equivalent for player i if the difference

$$\tilde{u}_i(x_1, \dots, x_n) - u_i(x_1, \dots, x_n) = c(x_{-i})$$ → constant depending only on x_{-i}

does not depend on their decision x_i but on the strategies of the other players.

Theorem

Finite games with diff equivalent payoffs have the same pure Nash equilibria.

Proof

By definition, diff-equivalent payoffs are s.t. $\forall x_i', x_i \in X_i$: $\tilde{u}_i(x_i', x_{-i}) - u_i(x_i', x_{-i}) = \tilde{u}_i(x_i, x_{-i}) - u_i(x_i, x_{-i}) \longrightarrow \Delta \tilde{u}_i(x_i', x_i, x_{-i}) = \Delta u_i(x_i', x_i, x_{-i})$

A profile $(x_1, \dots, x_n)$ is a pure NE iff the payoff increment when moving from x_i to any other x_i' are non positive : $\Delta u_i(x_i', x_i, x_{-i}) \leq 0$

Therefore, if two games have diff-equivalent payoffs, $\Delta \tilde{u}_i = \Delta u_i$, they also have the same NE as $\Delta \tilde{u}_i = \Delta u_i \leq 0$

$$\Delta f(a,b,c) = f(a,c) - f(b,c)$$

POTENTIAL GAME

A finite game with strategy sets X_i and payoffs $u_i : X \to \mathbb{R}$ is called potential game if it is diff-equivalent to a game with common payoffs, that is, there exists a potential function $p: X \to \mathbb{R}$ such that for each i, for every $x_{-i} \in X_{-i}$, and all $x_i', x_i \in X_i$ we have:

$$\Delta u_i(x_i', x_i, x_{-i}) = \Delta p(x_i', x_i, x_{-i})$$

ADVANTAGE: only one utility function.

COROLLARIES

- every finite potential game has at least one pure NE.
- in a finite potential game every best response iteration reaches a NE in finitely many steps.

ROUTING GAMES (slide 9)

Social Cost and Efficiency

- NE is not necessarily Pareto efficient.
- Need to quantify how bad a NE is.

→ The quality of a strategy profile $x = (x_1, \dots, x_n)$ is measured through a social cost function $x \mapsto C(x)$ where $C : X \to \mathbb{R}^+$. The smaller $C(x)$ the better the outcome $x \in X$.

- The benchmark is the minimal value that a benevolent social planner could achieve: $Opt = \min_{x \in X} C(x)$

For $x \in X$, the quotient $\frac{C(x)}{Opt}$ measures how far is x from being optimal. If $\gg$ then social loss. If $=1$ optimal

Price of Anarchy

$$PoA = \max_{\bar{x} \in NE} \frac{C(\bar{x})}{Opt}$$

Choosing the worst NE among the possible

Price of Stability

$$PoS = \min_{\bar{x} \in NE} \frac{C(\bar{x})}{Opt}$$

The best NE we can get if we want stability

COOPERATIVE GAMES

A cooperative game is a pair (N, V) where
players, utilities

$$V : \mathcal{P}(N) \longrightarrow \mathbb{R}^n \quad , \quad V(A) \subseteq \mathbb{R}^A$$

insieme delle parti

$|\mathcal{P}(N)| = 2^n$

coalition

utilities of coalition A

in realtà sarebbe $\mathbb{R}^n$, ma $\mathbb{R}^A$ sarebbe vettore dove solo elementi di A sono $\neq 0$

A transferable utility game TU game is a function
$$v : 2^N \longrightarrow \mathbb{R}$$
such that $v(\emptyset) = 0$.

NOTE: a TU game does not specify how to split the utility among the players.

A TU game is also a cooperative game:

$$V(A) = \{ x \in \mathbb{R}^A : \sum_{i \in A} x_i \leq v(A) \}$$

A bankrupcy game is defined by the triple $B = (N, c, E)$ where $N = \{1, \dots, n\}$ is the set of creditors $c = \{c_1, \dots, c_n\}$ where c_i represents the credit claimed by player i and E is the estate. The bankrupcy condition is then $E < \sum_{i \in N} c_i = C$

pessimistic: $v_p(S) = \max\left(0, E - \sum_{i \in N \setminus S} c_i\right)$, $S \subseteq N$

if there is some money left after others are served.

optimistic: $v_o(S) = \min\left(\sum_{i \in S} c_i, E\right)$, $S \subseteq N$

all the money they owe to S

Airport game

A group of airlines who flies N airplanes needs a new runway close to some city. The set of airplanes is partitioned into groups of similar size $N_1, \dots, N_k$ and to each N_j corresponds the cost c_j of the runway construction.

$$v(S) = \max\{ c_i : i \in S \} \longrightarrow \text{how to share total cost among different groups?}$$

Peer game

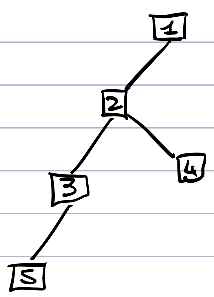
$N = \{1, \dots, n\}$ set of players and $T = (N, A)$ directed rooted tree. Each agent i has an individual potential v_i which represents the gain that player i can generate if all players at higher levels of the hierarchy cooperate with him.

For every $i \in N$, we denote by $S(i)$ the set of all agents in the unique directed path connecting 1 to i, i.e. the set of superiors of i.

A peer game is the game v such that

$$v(S) = \sum_{i \in N : S(i) \subseteq S} v_i \quad .$$

The question is how to divide the value among the players.



The set of TU games

Let $G(N)$ be the set of all cooperative games having N as set of players. Fix a list $S_1, \dots, S_{2^n-1}$ of coalitions.

A vector $(v_1, \dots, v_{2^n-1})$ represents a game, setting $v_i = v(S_i)$.

Thus $G(N)$ is isomorphic to $\mathbb{R}^{2^n-1}$

PROPOSITION

game

The set $\{u_A : A \subseteq N\}$ of the unanimity games u_A

$$u_A(T) = \begin{cases} 1 & \text{if } A \subseteq T \\ 0 & \text{otherwise} \end{cases}$$

is a basis for the space $G(N)$

A game is additive if

$$v(A \cup B) = v(A) + v(B) \qquad \text{for all } A \cap B = \emptyset$$

A game is Superadditive if

$$v(A \cup B) \geq v(A) + v(B) \qquad \text{for all } A \cap B = \emptyset$$

→ it means that it's convenient to form coalitions.

A simple game $v \in G$ is a game where

- $v(S) \in \{0, 1\}$ for every nonempty coalition S
- $A \subseteq C$ implies $v(A) \leq v(C)$
- $v(N) = 1$

NOTE: 1 means coalition A wins
0 means coalition A loses

it means that if A is a winning coalition then C is also a winning coalition, but not necessarily viceversa.

A minimal winning coalition A is a coalition in the simple game v s.t.

- $v(A) = 1$
- $B \subsetneq A$ implies $v(B) = 0$

SOLUTIONS

A solution vector for the game $v \in G(N)$ is a vector $\underbrace{(x_1, \dots, x_n)}_{\text{the utilities of all players}}$.

A solution concept for the set of games $G(N)$ is a multifunction $S : G(N) \to \mathbb{R}^n$

IMPUTATION

The solution $I : G(N) \to \mathbb{R}^n$ such that $x \in I(v)$ is an imputation if

- $x_i \geq v(\{i\})$ for all $i \in N$ EFFICIENCY: being in a coalition is at least as good as being alone
- $\sum_{i=1}^n x_i = \underbrace{v(N)}$

↳ utility when the coalition includes all players.

→ if $v(N) \geq \sum_i v(\{i\})$ the imputation is nonempty

→ if a game is additive then the imputation is $\{(v(1), \dots, v(n))\}$

PROPOSITION

The imputation set $I(v)$ is a polytope (the smallest closed convex set containing a finite number of pts)

- Efficiency is a mandatory requirement
- The imputation set is nonempty if the game is superadditive
- The imputation set lies in the hyperplane $H = \{x \in \mathbb{R}^n : \sum_{i=1}^n x_i = v(N)\}$ and it is bounded since $x_i \geq v(\{i\})$ for all $i \in N$.

CORE

this means all coalitions prefer I to the coalition

The core is the solution $C: G(N) \to \mathbb{R}^n$ such that:

$$\boxed{C(v) = \{x \in \mathbb{R}^n : \sum_{i=1}^{n} x_i = v(N) \ \wedge \ \sum_{i \in S} x_i \geq v(S) \ , \forall S \subseteq N\}}$$

PROPOSITION

- The core is a subset of the set of imputations.
- Imputations are efficient distributions of utilities accepted by all players individually.
- Core vectors are efficient distributions of utilities accepted by ALL coalitions.

Structure of the core

The core $C(v)$ is a polytope (i.e. the smallest closed convex set containing a finite # of pts)

PROPOSITION

The core reduces to the singleton $(v(\{1\}), \dots, v(\{n\}))$ if v is additive
The core of superadditive games can be empty.

Veto player

In a game v, a player i is a veto player if $v(A) = 0 \ \forall A$ s.t. $i \notin A$.

Theorem

Let v be a simple game. Then $C(v) \neq \emptyset$ iff there is at least one veto player.
When a veto player exists, the core is the closed convex polytope with extreme points the vectors $(0, \dots, 1, \dots, 0)$ where the 1 corresponds to a veto player.

Proof

If there is no veto player then $\forall i \in N$, $N \setminus \{i\}$ is a winning coalition (otherwise i would be a veto player)
Suppose $(x_1, \dots, x_n) \in C(v) \longrightarrow \sum_{j \neq i} x_j = 1, \forall i = 1, \dots, n$, since $x_i = 0$ and $V(N) = 1$

Summing up... $\sum_{i=1}^{n} \overbrace{\sum_{j \neq i} x_j}^{1} = (n-1) \sum_{j=1}^{n} x_j = n$, which is a contradiction since $\sum_{j=1}^{n} x_j = 1$.

Conversely, any imputation assigning zero to all the non-veto players is in the core, since $v(S) = 1$ implies all veto players belong to S.

Theorem - NON EMPTYNESS OF THE CORE

LP problem:
$$\min \sum_{i=1}^{n} x_i$$
$$\text{s.t.} \sum_{i \in S} x_i \geq v(S) \qquad \forall S \subseteq N \qquad \star$$

The LP problem above always has a nonempty set of solutions C. The core $C(v)$ is nonempty and $C(v) = C$ iff the optimal value of the LP is $v(N)$.

Note that the value V of the LP problem is $V \geq v(N)$, due to the constraint $\sum_{i=1}^{n} x_i \geq v(N)$; thus for every x fulfilling $\star$ it holds $\sum_{i=1}^{n} x_i \geq v(N)$

The LP problem $\mathcal{P}$ associated to the core problem has the following matrix form:

$$\begin{cases} \min \langle c, x \rangle \\ Ax \geq b \end{cases}$$

where $c = 1_N$, $b = (v(\{1\}), \dots, v(N))$
and A is a $2^n - 1 \times n$ matrix with the following features:
a) it is boolean
b) the 1s in row j are in correspondence with players in S_j.

The dual of the problem is of the form:

$$\begin{cases} \max \sum_{S \subseteq N} \lambda_S v(S) \\ \lambda_S \geq 0 \\ \sum_{S: i \in S \subseteq N} \lambda_S = 1 \quad \forall i \end{cases}$$

Since the primal has solutions, the fundamental duality theorem states that also the dual has solution, and there is no duality gap. Thus the core $C(v)$ is nonempty iff the value V of the dual problem is s.t. $V \leq v(N)$. It follows:
$C(v) \neq \emptyset$ iff every vector $(\lambda_S)_{S \subseteq N}$ fulfilling the conditions:
1) $\lambda_S \geq 0 \ \forall S \subseteq N$ and
2) $\sum_{S: i \in S \subseteq N} \lambda_S = 1 \quad \forall i \in N$

verifies also:

$$\sum_{S \subseteq N} \lambda_S v(S) \leq v(N)$$

Let v be some TU game. The excess of a coalition A over the imputation x is $e(A,x) = v(A) - \sum_{i \in A} x_i$ and can be intended as a measure of the dissatisfaction of the coalition A w.r.t. the assignment of the imputation x.
Note that an imputation x of the game v belongs to $C(v)$ iff $e(A,x) \leq 0 \ \forall A$.

The lexicographic vector attached to the imputation x is the $(2^n - 1)$-th dimensional vector $\theta(x)$ s.t.
$\theta_i(x) = e(A,x)$, for some $A \subseteq N$
$\theta_1(x) \geq \theta_2(x) \geq \dots \geq \theta_{2^n - 1}(x)$

The nucleolus solution is the solution $\nu: G(N) \to \mathbb{R}^n$ s.t. $\nu(v)$ is the set of imputations x s.t. $\theta(x) \leq_L \theta(y)$ for all y imputations of the game v.
Note that $x \leq_L y$ if $x = y$ or $\exists j \mid x_i = y_i \ \forall i < j$ and $x_j < y_j$.

For every TU game v with nonempty imputation set, the nucleolus $\nu(v)$ is a singleton

Theorem
Suppose v is s.t. $C(v) \neq \emptyset$. Then $\nu(v) \in C(v)$.

Proof
For all $x \in C(v)$, $\theta_1(x) \leq 0$.
Since the nucleolus minimizes the excess, we have $\theta_1(\nu(v)) \leq 0$.
Then $\nu(v)$ is in the core. ∎

The SHAPLEY value and power indices

Let $\phi : G(N) \to \mathbb{R}^m$ be a one point solution

Desirable properties:

EFFICIENCY * $\sum_{i \in N} \phi_i(v) = v(N)$, $\forall v \in G(N)$

SYMMETRY * if $v \in G(N)$ is a game s.t. $\forall A$ not containing i, j, $v(A \cup \{i\}) = v(A \cup \{j\})$ then $\phi_i(v) = \phi_j(v)$

NULL PLAYER * if $v \in G(N)$ and $i \in N$ s.t. $v(A) = v(A \cup \{i\})$ $\forall A$ then $\phi_i(v) = 0$

ADDITIVITY * $\phi(v+w) = \phi(v) + \phi(w)$ $\forall v, w \in G(N)$

Theorem - Shapley

Consider the following function $\sigma : G(N) \to \mathbb{R}^m$

$$\sigma_i = \sum_{S \in 2^{N \setminus \{i\}}} \underbrace{\frac{s!\,(m-s-1)!}{m!}}_{\text{probability of joining coalition } S \text{, with } |S| = s} \overbrace{[v(S \cup \{i\}) - v(S)]}^{\text{marginal contribution of player } i \text{ to coalition } S \cup \{i\}}$$

Then σ is the only function that satisfies the properties of efficiency, symmetry, null player and additivity.

Can be intended as the weighted sum of all marginal contributions of the players.

Proof

Efficiency: $\sum_{i=1}^{m} \sigma_i(v) = v(N)$

Consider the generic term $v(S \cup \{i\}) - v(S)$. The term $v(N)$ appears n times, once for every player, when $S = N \setminus \{i\}$.

Its coefficient is $\frac{(m-1)!\,(m-m)!}{m!} = \frac{1}{m}$

no other contributions

Consider now $T \neq N$, the term $v(T)$ appears both with positive and negative coefficients:

* the positive coefficient $\frac{(t-1)!\,(m-t)!}{m!}$ appears t times, one for every player $i \in S$, when $S = T \setminus \{i\}$: its contribution is $\frac{t!\,(m-t)!}{m!}$ ($t \cdot (t-1)! = t!$)

* the negative coefficient $-\frac{t!\,(m-t-1)!}{m!}$ appears $m-t$ times, one for every player $i \notin T$, when $S = T$: its contribution is $-\frac{t!\,(m-t)!}{m!}$

Thus in the sum $\sum_{i=1}^{m} \sum_{S \in 2^{N \setminus \{i\}}} \frac{s!\,(m-s-1)!}{m!} (v(S \cup \{i\}) - v(S))$

$v(N)$ appears with coefficient 1 and every $A \neq N$ appears with null coefficient.

Therefore $\sum_{i=1}^{m} \sigma_i(v) = v(N)$. ■

Symmetry: if v, i, j s.t. for every A not containing i, j $v(A \cup \{i\}) = v(A \cup \{j\})$ then $\sigma_i(v) = \sigma_j(v)$

Consider $\sigma_i(v) = \sum_{S \in 2^{N \setminus \{i,j\}}} \frac{s!\,(m-s-1)!}{m!} [v(S \cup \{i\}) - v(S)] + \sum_{S \in 2^{N \setminus \{i,j\}}} \frac{(s+1)!\,(m-s-2)!}{m!} [v(S \cup \{i,j\}) - v(S \cup \{j\})]$

$\|$

$\sigma_j(v) = \sum_{S \in 2^{N \setminus \{i,j\}}} \frac{s!\,(m-s-1)!}{m!} [v(S \cup \{j\}) - v(S)] + \sum_{S \in 2^{N \setminus \{i,j\}}} \frac{(s+1)!\,(m-s-2)!}{m!} [v(S \cup \{i,j\}) - v(S \cup \{i\})]$

■

Null player: marginality is $0 \to \sigma_i = 0$

Additivity: $v(S) = v_1(S) + v_2(S) \to \sigma(i) = \sigma_1(i) + \sigma_2(i) \Rightarrow \sum \frac{s!\,(m-s-1)!}{m!} [v(S \cup \{i\}) - v(S)] = \sum \frac{s!\,(m-s-1)!}{m!} [v_1(S \cup \{i\}) + v_2(S \cup \{i\}) - v_1(S) - v_2(S)] \ldots$

Uniqueness: 1) given a unanimity game u_A:

- players not belonging to A are null players: thus ϕ assigns 0 to them
- players in A are symmetric, so ϕ must assign the same amount to all.
- ϕ is efficient

2) ϕ is uniquely determined on the basis of $G(N)$ of the unanimity games

3) The same argument applies to the game $c \cdot u_A$, $c \in \mathbb{R}$

By the additivity axiom at most one function satisfies the properties.

In the case of simple games, the Shapley value becomes:

$$\sigma_i(v) = \sum_{A \in \mathcal{A}_i} \frac{a!\,(m-a-1)!}{m!}$$

where $\mathcal{A}_i$ is the set of coalitions A s.t. $i \notin A$, A is not winning, $A \cup \{i\}$ is winning.

Power indices for simple games

In simple games the Shapley value assumes also the meaning of measuring the fraction of power of every player.
To measure the relative power of the players in a simple game, the efficiency requirement is not mandatory and the way coalitions could form can be different from the case of the Shapley value

Probabilistic power index ψ on the set of simple games is

$$\psi_i(v) = \sum_{S \in 2^{N \setminus \{i\}}} p_i(S) \cdot m_i(v, S)$$, where p_i is a probability measure on $2^{N \setminus \{i\}}$

Semivalue A probabilistic power index such that $p_i(S) = p(s) \quad \forall i \in N, \; s = |S|$

These must hold:
* $p_s \geq 0$
* $\sum_{s=0}^{n-1} \binom{n-1}{s} p_s = 1$.

If $p_s > 0$ for all s, the semivalue is regular.

Examples:
- Shapley value
- Banzhaf value, $p_s = \frac{1}{2^{n-1}}$
- binomial value, $p_s = q^s (1-q)^{n-s-1}$
- marginal value, $p_s = 0$ for $s = 0, \dots, n-2$, $p_{n-1} = 1$
- dictatorial value, $p_s = 0$ for $s = 1, \dots, n-1$, $p_0 = 1$

Matching Problems

- Idea: form couples among two groups.
- Problem: what is the best definition of stable placements?
- Assumptions: there are two groups, same cardinality, each element has a ranking on the elements of the other group.

(Strict) Preference Relation

Let X be a set. A (strict) preference relation on X is a binary relation $\succ$ fulfilling, for all $x, y, z \in X$:

- $\rightarrow$ if $x \neq y$ either $x \succ y$ or $y \succ x$ COMPLETENESS
- $\rightarrow$ $x \not\succ x$ IRREFLEXIVITY
- $\rightarrow$ $x \succ y \wedge y \succ z$ imply $x \succ z$ TRANSITIVITY

Matching Problem

A matching problem is given by:

1) a natural number n (common cardinality of two sets A, B, of which elements are called women and men)
2) a set of preferences s.t. each woman has a preference relation over the set of men and conversely

A matching is a bijection between the two sets.

A pair man-woman (m, w) objects to the matching Λ if m and w both prefer each other to the person paired to them in the matching Λ.

A matching Λ is called stable provided there is no pair woman-man objecting to Λ.

Theorem
Every matching problem admits a stable matching.

Proof night by night

- every woman visits her most preferred choice
- every man chooses 1 woman among those
- every women not matched, visits her 2nd best choice
- every man chooses 1 woman among those + the one he already had, if any.

PROPERTIES

- Women go down along their preferences
- men go up along their preferences
- if a men is visited at stage r, then from stage $r+1$ on he will never be alone.
- the algorithm provides 2 matchings (one if women starts, one if men do)
- every woman can visit at most n men $\rightarrow n^2 \longrightarrow n^2 - n + 1$ b/c the first night all women are involved
- every men is visited at some stage $\longrightarrow$ no men remains alone.
- no women can be part of an objecting pair (therefore, since a pair is made up by (woman, man) there is no objecting pair) This is because women pick men in descending preference $\longrightarrow$ no way they can object.

- there are $n!$ possible matchings

Given two matchings Δ and Θ. We say $\Delta \geq_m \Theta$ if every man is either associated to the same woman in the two matchings or associated to a preferred woman in Δ (analogously for women $\Delta \geq_w \Theta$).

- $\Delta \geq_m \Delta$ REFLEXIVITY
- $\Delta \geq_m \Theta \wedge \Theta \geq_m \Lambda \implies \Delta \geq_m \Lambda$ TRANSITIVITY

Theorem

Let Δ and Θ be stable matchings. Then $\Delta \geq_m \Theta$ iff $\Theta \geq_w \Delta$

Proof

Suppose that $\Delta \geq_m \Theta$. Let $(a, A) \in \Delta$ and $(b, A) \in \Theta$.
We have to prove that $b \geq_A a$
Suppose that $(a, F) \in \Theta$. We have $A \geq_a F$ because $\Delta \geq_m \Theta$.
So $\{(a,F), (b,A)\} \subset \Theta$, but Θ is a stable matching
therefore $b \geq_A a$, otherwise the pair (a, A) would object.

Theorem

Let Λ_m be the men visiting matching and let Θ be another stable matching. Then
$$\Lambda_m \geq_m \Theta \geq_m \Lambda_w \qquad \Lambda_w \geq_w \Theta \geq_w \Lambda_m$$
Women visiting is the best algorithm for the women (same for men)

Proof

Let's prove that a woman cannot be rejected by a man available to her.
First day: suppose A is rejected by a in favor of B and that there is a stable set Δ s.t.
$$\{(a,A), (b,B)\} \subset \Delta$$
Then since $B >_a A$, then $b >_B a$. → Otherwise wouldn't be stable
But this is impossible since by assumption B is visiting a in the first day, thus a is her preferred man.

Suppose no woman was rejected by an available man the days $1 \ldots k-1$.
By contradiction, suppose A is rejected by a in favor of B on day k and that there is a stable set Δ s.t.
$$\{(a,A), (b,B)\} \subset \Delta.$$
Then since $B >_a A$ then $b >_B a$.
Since B is visiting a, but likes better b, then B visited b some day before and was rejected, against the inductive assumption.
Hence, for every stable Θ we have $\Lambda_w \geq_w \Theta$. To conclude it suffices to use symmetry between men and women.

Let Δ, Θ be two matchings. Define a new "matching" $\Delta \vee_w \Theta$ by pairing each woman to the preferred man between those paired to her in Δ and Θ.
It may happen that two women are paired with the same man.

Theorem (maybe ask for the proof to someone).

Let Δ, Θ be two stable matchings. Then $\Delta \vee_m \Theta$ is a stable matching.
Then $\vee_m$ provides a lattice structure (a partial order s.t. every two elements have a unique maximum and a unique minimum) to the set of stable matchings.

EXTENSIONS, see slides